Exercise 2.2.12

(A nonlinear resistor) Suppose the resistor in Example 2.2.2 is replaced by a nonlinear resistor. In other words, this resistor does not have a linear relation between voltage and current. Such nonlinearity arises in certain solid-state devices. Instead of $I_R = V/R$, suppose we have $I_R = g(V)$, where $g(V)$ has the shape shown in Figure 3.

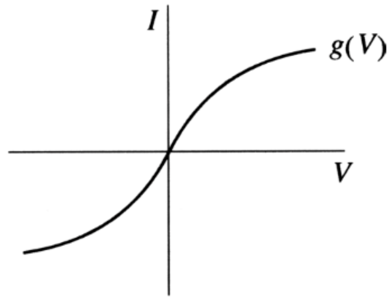


Figure 3

Redo Example 2.2.2 in this case. Derive the circuit equations, find all the fixed points, and analyze their stability. What qualitative effects does the nonlinearity introduce (if any)?

Solution

The aim in Example 2.2.2 is to analyze the RC circuit below.

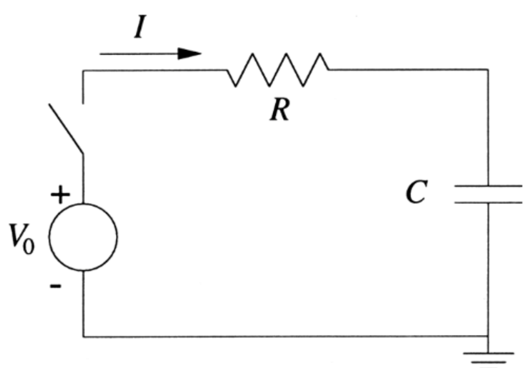


Figure 2.2.3

Apply Kirchhoff's voltage law for a clockwise current going through the closed circuit.

$$\sum_i V_i = 0$$

The voltage jumps from 0 to V_0 as the current passes through the battery, falls by V_R as the current passes through the resistor, and falls by V_C to 0 as the current passes through the capacitor.

$$+V_0 - V_R - V_C = 0$$

Substitute $V_R = V$ and $V_C = Q/C$.

$$V_0 - V - \frac{Q}{C} = 0$$

Observe that the graph of $I = g(V)$ is one-to-one (that is, every horizontal line intersects the curve exactly once). That means an inverse function g^{-1} exists: $V = g^{-1}(I)$.

$$V_0 - g^{-1}(I) - \frac{Q}{C} = 0$$

Solve for this inverse function.

$$g^{-1}(I) = V_0 - \frac{Q}{C}$$

Use the fact that current is the time derivative of charge.

$$g^{-1}(\dot{Q}) = V_0 - \frac{Q}{C}$$

Apply g to both sides.

$$\boxed{\dot{Q} = g\left(V_0 - \frac{Q}{C}\right)}$$

The fixed points occur where $\dot{Q} = 0$.

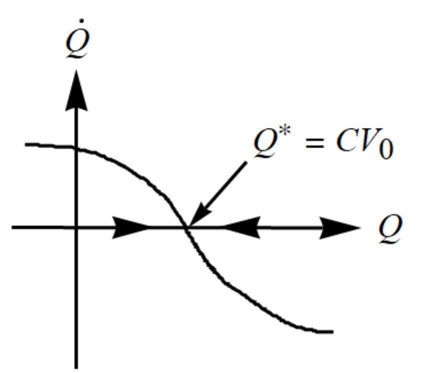
$$g\left(V_0 - \frac{Q^*}{C}\right) = 0$$

From the graph of $I = g(V)$, g is zero only when its argument is zero.

$$V_0 - \frac{Q^*}{C} = 0$$

$$\boxed{Q^* = CV_0}$$

Now plot $\dot{Q}$ versus Q to determine the stability of this fixed point. The minus sign in front of Q, reflects the graph of $g(V)$ about the vertical axis, the C in the denominator changes the scaling, and the V_0 translates the graph to the right.



When the function is negative the flow is to the left, and when the function is positive the flow is to the right. This makes $Q^* = CV_0$ a stable fixed point. The nonlinearity makes $\dot{Q}$ higher in magnitude around the fixed point than it would be if the resistor were linear, so equilibrium is reached faster. In other words, the nonlinear resistor makes the capacitor charge faster.